MATH329: Final Exam (May 10th, 1pm–3pm)

Name: ___________________________________

Give clear and concise arguments for each of your claims.
A standard 8 1/2 by 11 sheet of paper with student's notes (both sides) is allowed.

1. Consider the matrix

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 2 & 2 \\ 0 & 3 & 8 & 7 \\ 0 & 0 & 4 & 2 \end{bmatrix}.$$

   (a) Find the kernel of $\boldsymbol{A}$.

   (b) Find the image of $\boldsymbol{A}$.

   (c) What is the reduced row echelon form of matrix $\boldsymbol{B}$, where

$$\boldsymbol{B} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{A} \\ \boldsymbol{A} & \boldsymbol{A} \end{bmatrix},$$

   where $\boldsymbol{A}$ as above?

2. Let

$$\boldsymbol{A} = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}.$$

   Find $\boldsymbol{A}^{2004}$.

3. True or false (give reasons):

   (a) Every unitary matrix is normal.

   (b) A matrix is unitary if and only if it is invertible.

   (c) The sum of two self-adjoint operators is self-adjoint.

   (d) If all eigenvalues of a normal operator are 1, then the operator is identity.

4. Matrix $\boldsymbol{A}$ is called nilpotent if $\boldsymbol{A}^k = \boldsymbol{0}$ for some k. Prove that the only eigenvalue of a nilpotent matrix is zero.

5. Prove that a real $m \times n$ matrix $\boldsymbol{A}$ has rank 1 if and only if there are vectors $\boldsymbol{u} \in \mathbf{R}^m$ and $\boldsymbol{v} \in \mathbf{R}^n$ such that

$$\boldsymbol{u}\boldsymbol{v}^\top = \boldsymbol{A}.$$

6. Find the determinant of

$$\boldsymbol{A}_4 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

   by using any method you prefer. Find also determinants of the smaller matrices $\boldsymbol{A}_3$ and $\boldsymbol{A}_2$ with the same pattern of zeros on the diagonal and ones elsewhere.

   Compute $\det \boldsymbol{A}_n$.